

SANDIA REPORT

SAND97-2382 • UC-405

Unlimited Release

Printed February 1998

A Mathematical Basis for Automated Structured Grid Generation with Close Coupling to the Flow Solver

Daniel W. Barnette

Prepared by

Sandia National Laboratories

Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia is a multiprogram laboratory operated by Sandia Corporation,
a Lockheed Martin Company, for the United States Department of
Energy under Contract DE-AC04-94AL85000.

Approved for public release; further dissemination unlimited.



Sandia National Laboratories

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

NOTICE: This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government, any agency thereof, or any of their contractors or subcontractors. The views and opinions expressed herein do not necessarily state or reflect those of the United States Government, any agency thereof, or any of their contractors.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Prices available from (615) 576-8401, FTS 626-8401

Available to the public from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Rd
Springfield, VA 22161

NTIS price codes
Printed copy: A03
Microfiche copy: A01

**A Mathematical Basis
for Automated Structured Grid Generation
with Close Coupling to the Flow Solver**

Daniel W. Barnette
Parallel Computational Sciences Department
Sandia National Laboratories
P.O. Box 5800
Albuquerque, New Mexico 87185-1111

Abstract

The first two truncation error terms resulting from finite differencing the convection terms in the two-dimensional Navier-Stokes equations are examined for the purpose of constructing two-dimensional grid generation schemes. These schemes are constructed such that the resulting grid distributions drive the error terms to zero. Two sets of equations result, one for each error term, that show promise in generating grids that provide more accurate flow solutions and possibly faster convergence. One set results in an algebraic scheme that drives the first truncation term to zero, and the other a hyperbolic scheme that drives the second term to zero. Also discussed is the possibility of using the schemes in sequentially constructing a grid in an iterative algorithm involving the flow solver. In essence, the process is envisioned to generate not only a flow field solution but the grid as well, rendering the approach a 'hands-off' method for grid generation.

Acknowledgements

The author wishes to acknowledge the helpful discussions and insight given by Dr. Patrick Knupp of Sandia's Parallel Computing Sciences Department. Reviews and insights offered from, and discussions with, several colleagues outside of Sandia were also invaluable to the author as ideas were developing. These include Dr. John A. Benek of MicroCraft (Tullahoma, TN), Dr. C. Wayne Mastin of Nichols Research (Vicksburg, MS), and Profs. Joe F. Thompson and David Whitfield of Mississippi State University (Starkville, MS).

Contents

Introduction	6
Discussion of Convection Truncation Error Terms	6
Grid Generation Equations Derived from the Convection Truncation Error Terms	7
A. Derivation using $TE1=0$ equation	8
B. Derivation using $TE2=0$ equation	12
Proposed Method for an Automated Tightly-Coupled Flow-Solver/Grid-Generator	15
Summary	16
References	17
APPENDIX: Derivation of Governing Grid Equations for $TE2=0$	21
Figures	
1. Uniform grid in computational space...	18
2. Example of a clustered algebraic grid ...	19
3. Example of a clustered algebraic grid ...	20
Tables	
1. Grid boundary conditions.	9
2. Comparison of well-known relations ...	13

A Mathematical Basis for Automated Structured Grid Generation with Close Coupling to the Flow Solver

Introduction

The fact that grid generation and flow field calculations depend, or should depend, on each other has driven efforts in grid adaptation, embedding, and patching methods to “loosely” couple the flow field with the grid generation process. In this paper, an attempt is made to more tightly couple the grid generation and flow solution process by deriving two sets of grid generation equations from the first two convection truncation error terms in the discretized governing flow equations. Consideration of the first truncation error term leads to an algebraic grid generation scheme that is independent of the flow field. Manipulation of the second truncation error term leads to a set of grid generation equations directly involving flow quantities. Both sets are constructed such that the error terms are driven to zero. Benefits of such tight coupling and error minimization may include the ability to compute more accurate solutions as well as possibly faster convergence. Also, the resulting equations indicate the possibility that the grid and flow field can develop simultaneously, thus automating the grid generation process. This may result in less user interaction in the inherently complex and problematic process of grid generation.

A mathematical analysis on the two truncation error terms is presented below. Solution methods are discussed. Sample algebraic grid solutions are presented. Work in progress to solve the second set and form an automated grid generation algorithm is also discussed. Finally, a summary of the analysis is given.

Discussion of Convection Truncation Error Terms

Two-dimensional convection terms in the Navier-Stokes equations may be written in transformed computational space[1,2] as

$$\frac{1}{J} \left(\rho U \frac{\partial f}{\partial \xi} + \rho V \frac{\partial f}{\partial \eta} \right) \quad (1)$$

where f can be either the u or v covariant velocity, ρ is the density, the Jacobian of the transformation is given by

$$J = x_{\xi} y_{\eta} - y_{\xi} x_{\eta} \quad (2)$$

and U and V are the contravariant velocities

$$U = u y_{\eta} - v x_{\eta} \quad (3)$$

$$V = v x_{\xi} - u y_{\xi} \quad (4)$$

The independent variables ξ and η are the coordinate directions in computational space.

The truncation errors for convection terms are arrived at by using second-order upwind differencing for the convection terms and second-order central differencing for the metric coefficients.[3,4] The total truncation error may be expressed as

$$T_e = \rho u(T_x) + \rho v(T_y) = T_{E1} + T_{E2} + \text{Higher Order Terms} \quad (5)$$

where

$$T_{E1} = \frac{\rho}{2J} \left[\left(U_{\xi\xi\xi} + V_{\eta\eta\eta} \right) f_x + \left(U_{\xi\xi\xi} + V_{\eta\eta\eta} \right) f_y \right] \quad (6)$$

$$T_{E2} = \frac{\rho}{J} \left\{ \left(U_{\xi\xi\xi} + V_{\eta\eta\eta} \right) f_{xx} + \left[U \left(y_{\xi\xi\xi} + x_{\xi\xi\xi} \right) + V \left(y_{\eta\eta\eta} + x_{\eta\eta\eta} \right) \right] f_{xy} + \left(U_{\xi\xi\xi} + V_{\eta\eta\eta} \right) f_{yy} \right\}. \quad (7)$$

Note that grouping of terms is dependent on the order of the derivative 'f'. For the present, derivatives of f are taken in physical space so that the equations may be written in a more compact form. They will be transformed to computational space where necessary. Higher order truncation error terms than these may be considered, but gaining useful information from them as can be done with the above terms appears to be an extremely arduous task.

Lee and Tsuei[3] extensively analyze Eqs. 6-7 and show that

1. T_{E1} does not contain terms representing the grid size and is of zero-order accuracy; hence, if this error term is dominant, grid refinement will not ensure error reduction; is zero for a uniform parallel grid
2. T_{E2} may dominate if grid expansion ratios are much larger or much smaller than unity, resulting in significant zero-order errors; is zero for a uniform parallel grid
3. Truncation errors can be reduced if the velocity vector is aligned with one of the grid lines; and
4. Grid orthogonality does not guarantee that truncation error can be minimized.

The question arises as to whether one can arrive at equation sets that force T_{E1} and T_{E2} to be identically zero throughout the solution process and which can be used to establish an appropriate grid point distribution. In general, the terms will not be zeroed by the solution of the flow field, for that is what is solved in the lower order terms. It will be shown below that such equations may be derived and that such grid point distributions do exist. Also, it will be proposed that the equations coupled to the flow solver may be used to construct the grid by iterating back and forth between the flow solver and grid generation algorithms.

Grid Generation Equations Derived from the Convection Truncation Error Terms

This sections deals with the derivations of the grid generation equations. Discussed in Subsection A are details associated with setting the first truncation error term to zero, *i.e.*, $T_{E1}=0$. The intent of this section is to derive an algebraic grid system that is uncoupled to the flow solver and which drives TE1 to zero. Simple two-dimensional examples are given to illustrate the results.

Discussed in Subsection B are the grid generation equations associated with setting $T_{E2}=0$. The resulting equations are completely coupled to the flow solver through velocity gradients. They are classified as 2nd-order, linear, hyperbolic, partial differential equations with variable coefficients. An exact solution process is known and will be discussed, but the author has not yet done the analysis.

The algebraic results derived next, and the system of partial differential equations derived in Section B below, appear to form the basis of a procedure whereby the grids can be generated simultaneously with the development of the flow field. This concept is discussed later and will be the focus of future research.

A. Derivation using $T_{EI}=0$ equation

Grid generation equations derived by setting $T_{EI}=0$ begins by setting the coefficients of f_x and f_y to zero independently in Eq. 6. The resulting equations are

$$Ux_{\xi\xi\xi} + Vx_{\eta\eta\eta} = 0 \quad (8)$$

$$Uy_{\xi\xi\xi} + Vy_{\eta\eta\eta} = 0. \quad (9)$$

If a satisfactory distribution in x and y can be constructed that satisfies the above equations, T_{EI} will be zero. Rewriting the equations in matrix form yields

$$\begin{bmatrix} x_{\xi\xi\xi} & x_{\eta\eta\eta} \\ y_{\xi\xi\xi} & y_{\eta\eta\eta} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

from which a non-trivial solution can be obtained only if the determinant of the matrix is zero, *i.e.*,

$$x_{\xi\xi\xi}y_{\eta\eta\eta} - x_{\eta\eta\eta}y_{\xi\xi\xi} = 0. \quad (11)$$

The matrix elements must be determined or specified. With four unknowns and two equations, two of the matrix elements must be specified unless all terms are set to zero. For example only, consider grid distributions that can be constructed such that $x_{\xi\xi\xi}$ and $x_{\eta\eta\eta}$ are zero, thus satisfying Eq. 11. In this case, $y_{\eta\eta\eta}$ would be determined by Eq. 9 with $y_{\xi\xi\xi}$ specified, or *vice versa*. This would render part of the analysis closely coupled to the flow solver (a more sophisticated coupling will be presented in Subsection B). The equations governing grid generation would then be

$$x_{\xi\xi\xi} = x_{\eta\eta\eta} = 0 \quad (12)$$

and

$$y_{\eta\eta\eta} = -\frac{U}{V}y_{\xi\xi\xi}. \quad (13)$$

Details and problems of implementing these equations during the flow solver computation will not be discussed here. The approach taken in this section, consistent with the goal of deriving an algebraic set of equations, is to set all grid derivatives to zero, viz.,

$$x_{\xi\xi\xi} = x_{\eta\eta\eta} = y_{\xi\xi\xi} = y_{\eta\eta\eta} = 0. \quad (14)$$

Integration of $x_{\xi\xi\xi}$ leads to the generic form of the solution in ξ as

$$x_1(\xi, \eta) = \frac{F_1(\eta)(\xi-1)^2}{2} + F_2(\eta)(\xi-1) + F_3(\eta) \quad (15)$$

where $F_1(\eta)$, $F_2(\eta)$, and $F_3(\eta)$ are functions of η only. Note that the quantity $(\xi-1)$ is used in place of ξ with no loss in generality. Integrating $x_{\eta\eta\eta}$ leads to the generic form in η as

$$x_2(\xi, \eta) = \frac{G_1(\xi)(\eta-1)^2}{2} + G_2(\xi)(\eta-1) + G_3(\xi). \quad (16)$$

Table 1. Grid boundary conditions.

Computational Coordinates	Physical Coordinates
At $\xi=1, 1 \leq \eta \leq \text{KMAX}$:	$x=x_L(\eta), y=y_L(\eta)$
At $\xi=\text{JMAX}, 1 \leq \eta \leq \text{KMAX}$:	$x=x_R(\eta), y=y_R(\eta)$
At $\eta=1, 1 \leq \xi \leq \text{JMAX}$:	$x=x_B(\xi), y=y_B(\xi)$
At $\eta=\text{KMAX}, 1 \leq \xi \leq \text{JMAX}$:	$x=x_T(\xi), y=y_T(\xi)$
At $\eta=1, 1 \leq \xi \leq \text{JMAX}$: (Condition of orthogonality imposed at the Bottom boundary; first grid point spacing ΔS specified by user; could also be imposed at the Top boundary with slight modifications)	$\frac{\partial x}{\partial \eta} = \frac{-\Delta S \cdot y'_B}{\left[(x'_B)^2 + (y'_B)^2 \right]^{1/2}}; \quad \frac{\partial y}{\partial \eta} = \frac{\Delta S \cdot x'_B}{\left[(x'_B)^2 + (y'_B)^2 \right]^{1/2}}$
At $\xi=1, 1 \leq \eta \leq \text{KMAX}$: (Condition of orthogonality imposed at the Left boundary; first grid point spacing ΔN specified by user; could also be imposed at the Right boundary with slight modifications) OR At $\xi = \xi_M, 1 \leq \eta \leq \text{KMAX}$, where $1 < \xi_M < \text{JMAX}$ (Specified interior line; analysis not shown)	$\frac{\partial x}{\partial \xi} = \frac{\Delta N \cdot y'_L}{\left[(x'_L)^2 + (y'_L)^2 \right]^{1/2}}; \quad \frac{\partial y}{\partial \xi} = \frac{-\Delta N \cdot x'_L}{\left[(x'_L)^2 + (y'_L)^2 \right]^{1/2}}$ $x=x_M(\xi_M), y=y_M(\xi_M)$

Eqs. 15 and 16 respectively represent a family of η and ξ curves. Each family has three ‘constants’ of integration, all of which can be determined from the boundary conditions listed in Table 1 for an arbitrary grid.

The subscripts L, R, B, and T for x and y correspond to the Left, Right, Bottom, and Top of the grid as shown in Fig. 1. Indices $(\xi, \eta)=(1, 1)$ correspond to the point at which the Left and Bottom grid boundaries meet. Also note that $x'_B \equiv dx_B(\xi)/d\xi$, $y'_B \equiv dy_B(\xi)/d\xi$, *etc.* The terms ΔS and ΔN specify the distance of the first grid point spacing off either the bottom or left boundary, respectively. Evaluating the constants in Eqs. 15 and 16 with these generic boundary conditions leads to the equations

$$\begin{aligned}
 x_1(\xi, \eta) &= \left[\frac{\xi - 1}{JMAX - 1} \right]^2 x_R(\eta) + \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] x_L(\eta) \\
 &\quad + \frac{\Delta N \cdot y'_L}{\left[(x'_L)^2 + (y'_L)^2 \right]^{1/2}} (\xi - 1) \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right) \right] \\
 &\equiv \Omega_1(\xi) \cdot x_R(\eta) + \Omega_2(\xi) \cdot x_L(\eta) + \Omega_3(\xi) \cdot \frac{\Delta N \cdot y'_L}{\left[(x'_L)^2 + (y'_L)^2 \right]^{1/2}}
 \end{aligned} \tag{17}$$

and

$$\begin{aligned}
 x_2(\xi, \eta) &= \left[\frac{\eta - 1}{KMAX - 1} \right]^2 x_T(\xi) + \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right)^2 \right] x_B(\xi) \\
 &\quad - \frac{\Delta S \cdot y'_B}{\left[(x'_B)^2 + (y'_B)^2 \right]^{1/2}} (\eta - 1) \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right) \right] \\
 &\equiv \Gamma_1(\eta) \cdot x_T(\xi) + \Gamma_2(\eta) \cdot x_B(\xi) + \Gamma_3(\eta) \cdot \frac{-\Delta S \cdot y'_B}{\left[(x'_B)^2 + (y'_B)^2 \right]^{1/2}}
 \end{aligned} \tag{18}$$

where Ω_1 , Γ_1 , *etc.*, are the coefficients of $x_R(\eta)$, $x_T(\xi)$, *etc.*, respectively.

The family of curves represented by Eqs. 17 and 18 can be combined to interpolate the entire region and form a computational mesh. The two equations are combined using Boolean Sum projectors.[5] The final equation is given by

$$x(\xi, \eta) = x_1(\xi, \eta) + x_2(\xi, \eta)$$

$$\left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right)^2 \right] \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] x_B(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 x_B(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] x'_B(1) \Big\}$$

$$\begin{aligned}
& - \left[\frac{\eta - 1}{KMAX - 1} \right]^2 \left\{ \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] x_T(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 x_T(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] x_T'(1) \right\} \\
& - (\eta - 1) \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right) \right] \\
& \cdot \left\{ \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] x_L'(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 x_R'(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] \frac{\partial}{\partial \eta} \left[\frac{\partial x_L}{\partial \xi} \right]_{\xi = \eta = 1} \right\}.
\end{aligned}$$

Likewise, the analysis for y leads to the equations

$$y_1(\xi, \eta) = \left[\frac{\xi - 1}{JMAX - 1} \right]^2 y_R(\eta) + \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] y_L(\eta) \quad (19)$$

$$+ \frac{-\Delta N \cdot x_L'}{\left[\left(x_L' \right)^2 + \left(y_L' \right)^2 \right]^{1/2}} (\xi - 1) \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right) \right]$$

$$\equiv \Phi_1(\xi) \cdot x_R(\eta) + \Phi_2(\xi) \cdot x_L(\eta) + \Phi_3 \cdot \frac{-\Delta N \cdot x_L'}{\left[\left(x_L' \right)^2 + \left(y_L' \right)^2 \right]^{1/2}} \quad (20)$$

and

$$\begin{aligned}
y_2(\xi, \eta) &= \left[\frac{\eta - 1}{KMAX - 1} \right]^2 y_T(\xi) + \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right)^2 \right] y_B(\xi) \\
&+ \frac{\Delta S \cdot x_B'}{\left[\left(x_B' \right)^2 + \left(y_B' \right)^2 \right]^{1/2}} (\eta - 1) \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right) \right] \\
&\equiv \Lambda_1(\eta) \cdot y_T(\xi) + \Lambda_2(\eta) \cdot y_B(\xi) + \Lambda_3(\eta) \quad (21)
\end{aligned}$$

where Φ_1 , Λ_1 , *etc.*, are the coefficients of $y_R(\eta)$, $y_T(\xi)$, *etc.*, respectively. Combining Eqs. 20 and 21 using Boolean Sum projectors yields

$$\begin{aligned}
y(\xi, \eta) &= y_1(\xi, \eta) + y_2(\xi, \eta) \\
&- \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right)^2 \right] \left\{ \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] y_B(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 y_B(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] y_B'(1) \right\} \\
&- \left[\frac{\eta - 1}{KMAX - 1} \right]^2 \left\{ \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] y_T(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 y_T(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] y_T'(1) \right\} \\
&- (\eta - 1) \left[1 - \left(\frac{\eta - 1}{KMAX - 1} \right) \right] \\
&\cdot \left\{ \left[1 - \left(\frac{\xi - 1}{JMAX - 1} \right)^2 \right] y_L'(1) + \left[\frac{\xi - 1}{JMAX - 1} \right]^2 y_R'(JMAX) + (\xi - 1) \left[1 - \frac{\xi - 1}{JMAX - 1} \right] \frac{\partial}{\partial \eta} \left[\frac{\partial y}{\partial \xi} \right]_{\xi = \eta = 1} \right\}
\end{aligned} \tag{22}$$

The above grid distribution ensures $T_{E1}=0$ will be maintained. Some examples of grids generated using the above equations are presented in Figs. 2-3. The purpose in presenting the grids is to illustrate that intuitively reasonable grids can be constructed from the error analysis. These grids are neither coupled to the flow solver nor have they been run through a flow solver to investigate their effects on the solution process.

B. Derivation using $T_{E2}=0$ equation

The approach used to derive the desired equations from setting $T_{E2}=0$ is similar to that used for $T_{E1}=0$, above. However, the analysis is significantly more complicated since T_{E2} contains more derivatives and a cross-derivative term. The focus of the derivation will be to arrive at equations for a grid distribution that are dependent on the flow solution, hence coupling the flow solver with the grid generation process.

The analysis begins by setting Eq. 7 to zero. The equation is then rearranged to have factors that multiply the contravariant velocity terms U and V . This yields

$$\begin{aligned}
U \left[x_{\xi} x_{\xi\xi} f_{xx} + \left(y_{\xi} x_{\xi\xi} + x_{\xi} y_{\xi\xi} \right) f_{xy} + y_{\xi} y_{\xi\xi} f_{yy} \right] \\
+ V \left[x_{\eta} x_{\eta\eta} f_{xx} + \left(y_{\eta} x_{\eta\eta} + x_{\eta} y_{\eta\eta} \right) f_{xy} + y_{\eta} y_{\eta\eta} f_{yy} \right] = 0
\end{aligned} \tag{23}$$

Setting the coefficients of U and V to zero and rearranging gives

$$\left(x_{\xi} f_{xx} + y_{\xi} f_{xy} \right) x_{\xi\xi} + \left(x_{\xi} f_{xy} + y_{\xi} f_{yy} \right) y_{\xi\xi} = 0. \tag{24}$$

and

$$\left(x_{\eta} f_{xx} + y_{\eta} f_{xy} \right) x_{\eta\eta} + \left(x_{\eta} f_{xy} + y_{\eta} f_{yy} \right) y_{\eta\eta} = 0. \tag{25}$$

Working for the moment with Eq. 24, we again set the coefficients to zero which gives the two equations

$$x_{\xi} f_{xx} + y_{\xi} f_{xy} = 0 \quad (26)$$

and

$$x_{\xi} f_{xy} + y_{\xi} f_{yy} = 0. \quad (27)$$

Multiplying Eq. 26 by x_{ξ} , Eq. 27 by y_{ξ} , and subtracting removes the cross-derivative term. The result is

$$x_{\xi}^2 f_{xx} - y_{\xi}^2 f_{yy} = 0. \quad (28)$$

The analogous result for Eq. 25 is

$$x_{\eta}^2 f_{xx} - y_{\eta}^2 f_{yy} = 0. \quad (29)$$

In matrix form, the equations are written as

$$\begin{bmatrix} x_{\xi}^2 & -y_{\xi}^2 \\ x_{\eta}^2 & -y_{\eta}^2 \end{bmatrix} \begin{Bmatrix} f_{xx} \\ f_{yy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (30)$$

A non-trivial solution is arrived at by setting the determinant of the matrix to zero. This yields

$$x_{\xi}^2 y_{\eta}^2 - x_{\eta}^2 y_{\xi}^2 = (x_{\xi} y_{\eta} + x_{\eta} y_{\xi})(x_{\xi} y_{\eta} - x_{\eta} y_{\xi}) = (x_{\xi} y_{\eta} + x_{\eta} y_{\xi}) J = 0 \quad (31)$$

where the Jacobian J is given by Eq. 2. This equation bears close attention. Since it represents the grid cell area, the Jacobian of the transformation J cannot be zero. This leaves the coefficient of the Jacobian to be zero. It is instructive to compare this coefficient with the equation for the Jacobian, Eq. 2, and the well-known condition for orthogonality of grid lines, $\nabla \xi \bullet \nabla \eta = 0$, in the following table.

Table 2. Comparison of well-known relations with ‘Mystery Equation.’

Transformation Jacobian, Eq. 2:	$x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \neq 0$
Condition of Orthogonality:	$x_{\xi} x_{\eta} + y_{\xi} y_{\eta} = 0$
Mystery Equation, Eq. 31:	$x_{\xi} y_{\eta} + x_{\eta} y_{\xi} = 0$

Eq. 31 is listed as the ‘Mystery Equation’ since the author does not, at this time, understand its physical significance. However, this equation in its biquadratic form given in Eq. 31 is crucial to the analysis that follows. Without this equation, it is doubtful whether a grid distribution dependent on the flow solution could be arrived at that satisfies $T_{E2}=0$.

Since Eqs. 28 and 29 were used to arrive at Eq. 31, the proper set of equations to be solved involves one of Eqs. 28 or 29, and Eq. 31. Attention is now focused on Eq. 28. The equation is valid for $f=u$ or $f=v$, and both values are used below, along with Eq. 31. Setting $f=u$, transforming the f derivatives to the computational (ξ, η) plane, and regrouping terms as coefficients of the second derivative metrics result in the equation

$$\left[A_1\right]x_{\xi\xi} + \left[B_1\right]x_{\xi\eta} + \left[C_1\right]x_{\eta\eta} + \left[D_1\right]y_{\xi\xi} + \left[E_1\right]y_{\xi\eta} + \left[F_1\right]y_{\eta\eta} = RHS_1 \quad (32)$$

where the terms in brackets are shown in detail in the Appendix. It is also shown in the Appendix that, with the help of Eq. 31 (the ‘Mystery Equation’), every bracketed term in Eq. 32 is zero except for those multiplying the cross-derivatives $x_{\xi\eta}$ and $y_{\xi\eta}$, and the right hand side term. The term RHS_1 is also greatly simplified with the help of the Mystery Equation. The resultant equation is

$$u_x x_{\xi\eta} + u_y y_{\xi\eta} = u_{\xi\eta} \quad (33)$$

and the corresponding result for $f=v$ is

$$v_x x_{\xi\eta} + v_y y_{\xi\eta} = v_{\xi\eta} \quad (34)$$

These two equations allow us to solve for $x_{\xi\eta}$ and $y_{\xi\eta}$. Hence

$$x_{\xi\eta} = \Psi_1 x_{\xi} + \Psi_2 x_{\eta} \quad (35)$$

$$y_{\xi\eta} = \Psi_1 y_{\xi} + \Psi_2 y_{\eta} \quad (36)$$

where the velocity derivatives have been expressed in computational space and the coefficients represented by ψ are given by

$$\Psi_1 = \frac{v_{\eta} u_{\xi\eta} - u_{\eta} v_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}} \quad (37)$$

$$\Psi_2 = \frac{u_{\xi} v_{\xi\eta} - v_{\xi} u_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}}. \quad (38)$$

The cross-derivative terms that appear in Eqs. 35 and 36 can be a nuisance to constructing stable finite difference algorithms. This is due to their necessarily ‘wide’ stencil which degrades diagonal dominance for implicit schemes.[6] For an interesting discussion on how cross-derivatives affect weather prediction, for example, and the methods used to circumvent problems, confer Ames.[7]

In the author’s estimation, the reduction in terms, and in complexity, from Eq. 32 to that of Eqs. 35 and 36 is surprising. Just as surprising is the fact that the resultant equations have an exact solution form.[8] The solution is non-trivial and is being studied by the author. Finite difference methods are also being examined. Due to time constraints, the author has not developed either of these avenues.

Proposed Method for an Automated Tightly-Coupled Flow-Solver/Grid-Generator

It was stated in the introduction that one of the objectives of this report is to describe an algorithm for automated grid generation. Convection truncation error analyses have provided equations that can be used to develop such an algorithm. One set of equations for constructing grids is uncoupled from the flow solver; the other set, coupled. This permits an initial grid to be generated that ensures the leading convection truncation error term is zero. The initial grid is assumed to be of minimal construction and rather arbitrarily placed, with say five grid points off the wall. Once a flow solver has been run on the initial grid for several time steps and possibly some prominent features of the flow have begun to emerge, the non-converged solution can be fed into the coupled grid generation equations to more intelligently redistribute the grid points and possibly enhance flow field convergence[9]. This redistribution ensures the second leading truncation error term to be zero. A layer of grid points is then added to the previous outer layer of grid points, the process of which is discussed below. The algorithm then iterates between the flow solver and the grid construction until the flow solver indicates convergence and no more outer-layer grid points need be added.

The process of intelligently adding more grid points to the previous grid's outer layer is the method by which the grid is constructed, or built up. This allows the user to begin the flow solution with nothing more than the surface grid describing the geometry about which the flow field is to be calculated, and an initial coarse grid rather arbitrarily constructed with the use of algebraic methods such as those given by Eqs. 19 and 22. The equations used to construct, or add, the new outer layer are given by Eqs. 35 and 36. These equations are hyperbolic. A proper finite difference stencil allows the equations to predict a grid level, *i.e.* the new outer level, above those points at which the velocities and their gradients have been determined from the previous iteration. Development of this process appears to be straightforward.

Caveats abound in the development of any new method, and this method is no exception. One caveat is particularly noteworthy. If it is desired to use the approach presented herein to construct entirely algebraic grids for viscous dominated problems, the user will find it futile to use Eqs. 19 and 22. A quick glance at Eq. 17 will show why. It is likely that the first term after the equal sign will dominate the third term involving the user-specified spacing. This implies that the wall spacing as specified by the user may be very different than what actually occurs in the final grid. This should not be surprising, however, since the grid generation equations were derived from the convection error terms, not the diffusive terms dominant in the near-wall region. To counter this, one may consider using a highly clustered point distribution, such as arc-length blended transfinite interpolation, in the near-wall, viscous dominated region. This grid would then be matched at appropriate points with an algebraic grid generated using the methods presented herein. An alternative approach may be to derive a grid generation scheme based on the diffusive truncation error terms whose grid point distribution could then be blended with the equations presented above. The author has not examined the feasibility of either approach.

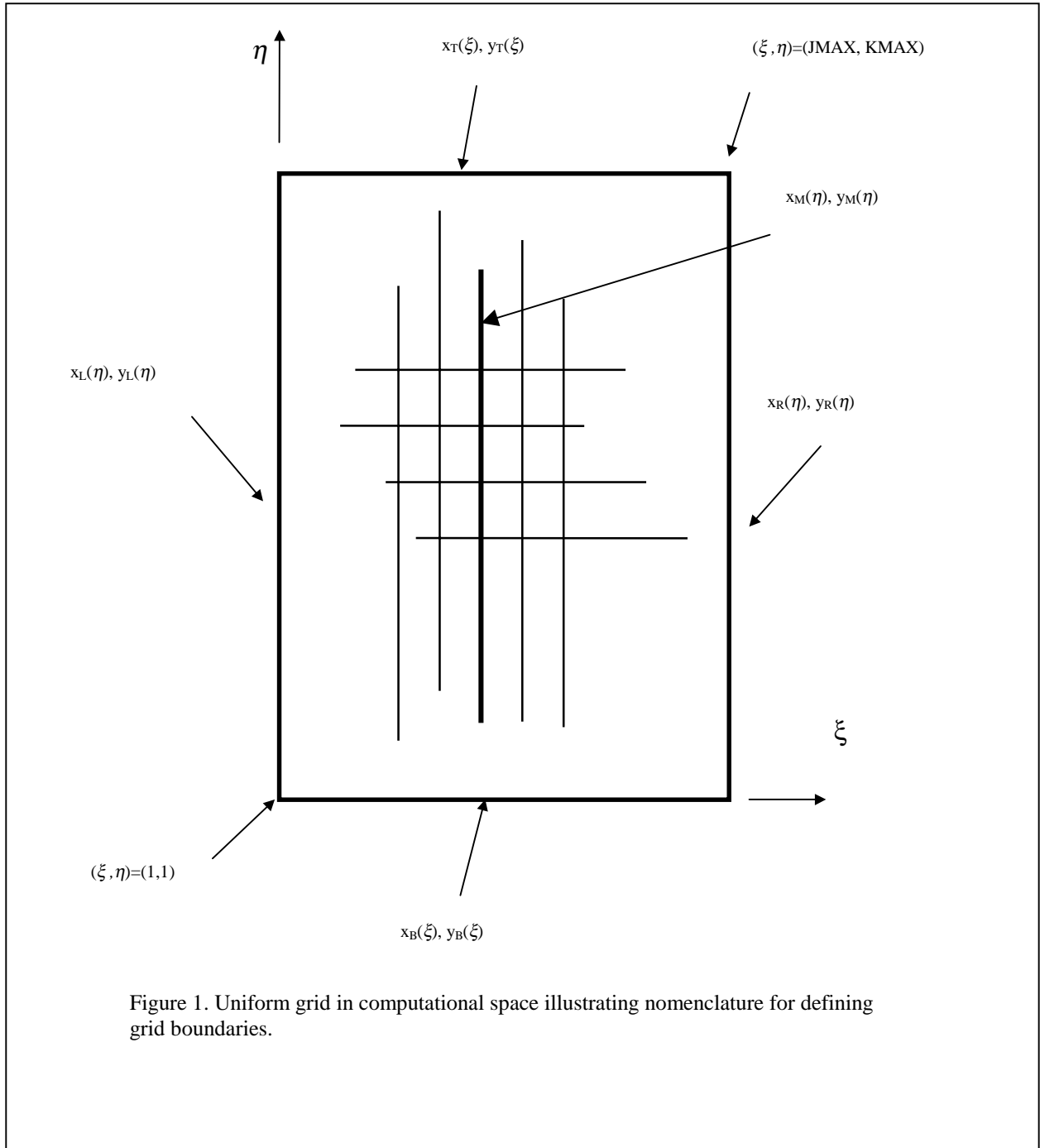
Summary

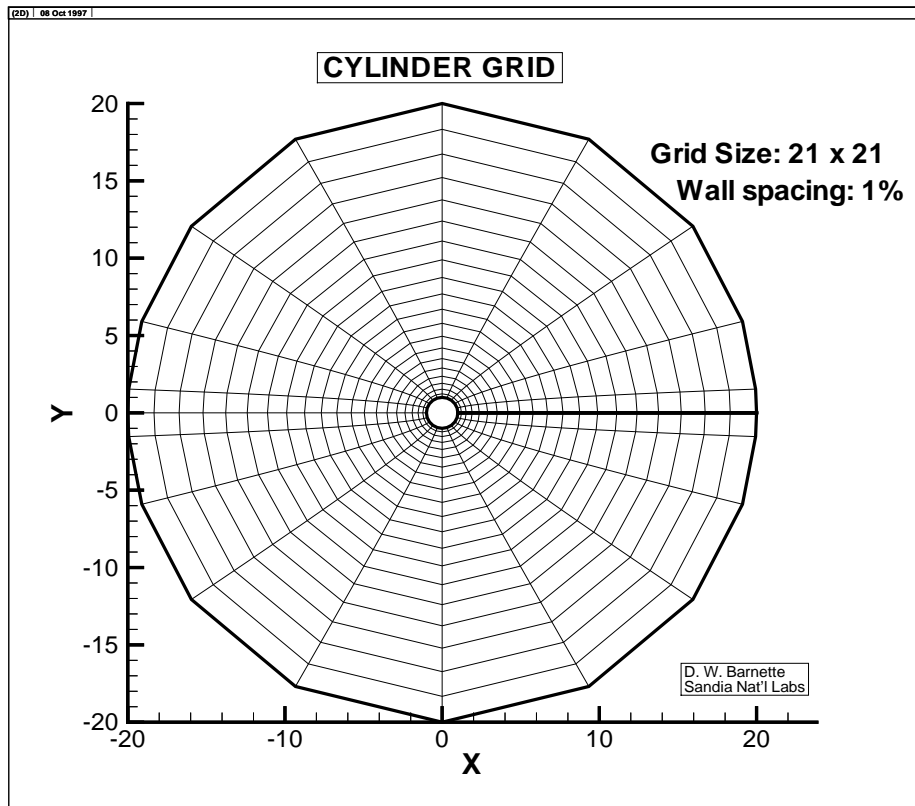
A mathematical basis has been presented for automated grid construction that is tightly coupled to the flow solver. The method is derived from equations whose solutions drive the first two leading convection truncation error terms to zero. An algorithm is also discussed whereby the equations may be used to automate grid construction that is tightly coupled with the flow solver. It is envisioned that nothing is specified about the grid being generated by the coupled flow solver other than the user-generated surface grid. Hence, the number of points as well as their distribution is not pre-determined. This approach differs substantially from the multi-grid or adaptive methods in that the grid is 'grown' in increments away from the relevant geometry. Grid points are added, and old points relocated, by utilizing feedback from the flow solver on the previously constructed, coarser grid.

The approach described above is novel, and it is unknown to the author if other researchers have attempted such a risky venture. However, if successful, the proposed algorithm will be extremely significant to computational algorithms for solving discretized governing equations. Many details need to be worked through, including the development of the approach in three dimensions, the effect of various flow solver differencing schemes on the form of the grid generation equations, and exactly how grids that drive truncation error terms to zero affect the flow solver's convergence rates and solution accuracy. Though much work remains to be done, the foundation of the analysis appears strong. The author hopes to soon have some practical applications and demonstrations of the power of the method.

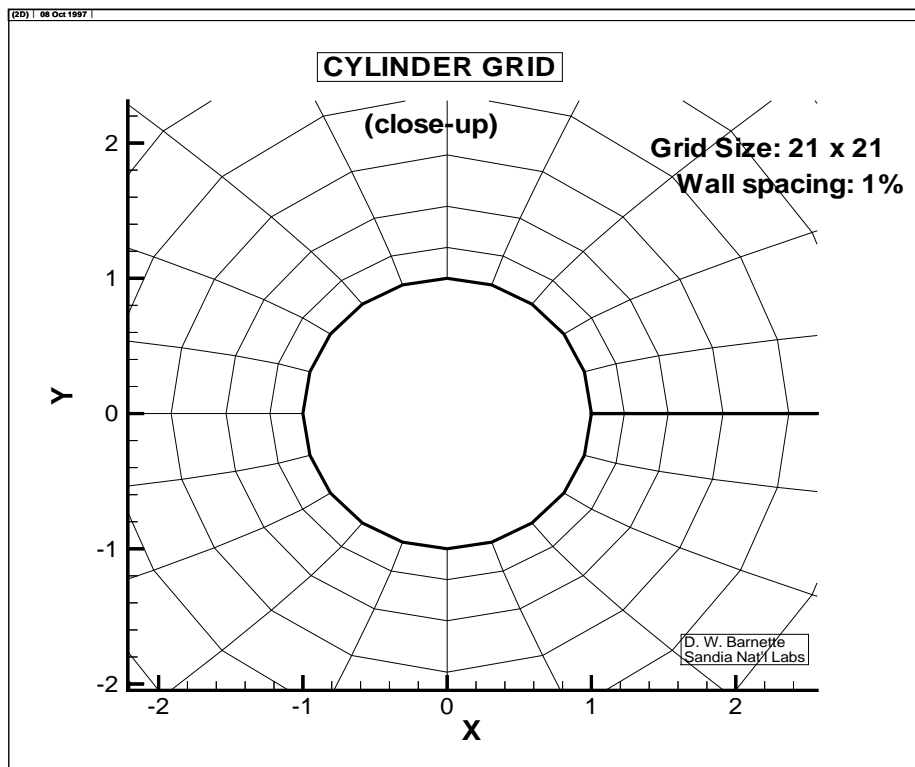
References

1. K. A. Hoffman and S. T. Chiang, Computational Fluid dynamics for Engineers, Vols. I and II, Engineering Education System, Kansas, 1993.
2. D. A. Anderson, J. C. Tannehill, and R. H. Pletcher, Computational Fluid Mechanics and Heat Transfer, Hemisphere Publishing Corporation, New York, 1984, especially Ch. 10.
3. D. Lee and Y. M. Tsuei, "A Formula for Estimation of Truncation Errors of Convection Terms in a Curvilinear Coordinate System," *Journal of Computational Physics* 98, pp. 90-100, 1992.
4. J. F. Thompson, Z. U. A. Warsi, and C. W. Mastin, Numerical Grid Generation, Foundations and Application, Elsevier Science Publishing Company, New York, 1985 (Chap. 5).
5. P. Knupp and S. Steinberg, Fundamentals of Grid Generation, CRC Press, Florida, 1997, pp. 92-94.
6. J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics, Springer, New York, 1996, p. 213.
7. W. F. Ames, Numerical Methods for Partial Differential Equations, 2nd edition, Academic Press, New York, 1977.
8. C. R. Chester, Techniques in Partial Differential Equations, McGraw-Hill, New York, 1971, pp. 222-229.
9. Private discussions with Dr. John A. Benek, Micro Craft, Tullahoma, Tennessee.
10. D. W. Barnette, "A User's Guide for BREAKUP: A Computer Code for Parallelizing the Overset Grid Approach," SAND98-0701, Sandia National Laboratories, Albuquerque, New Mexico, April 1998.



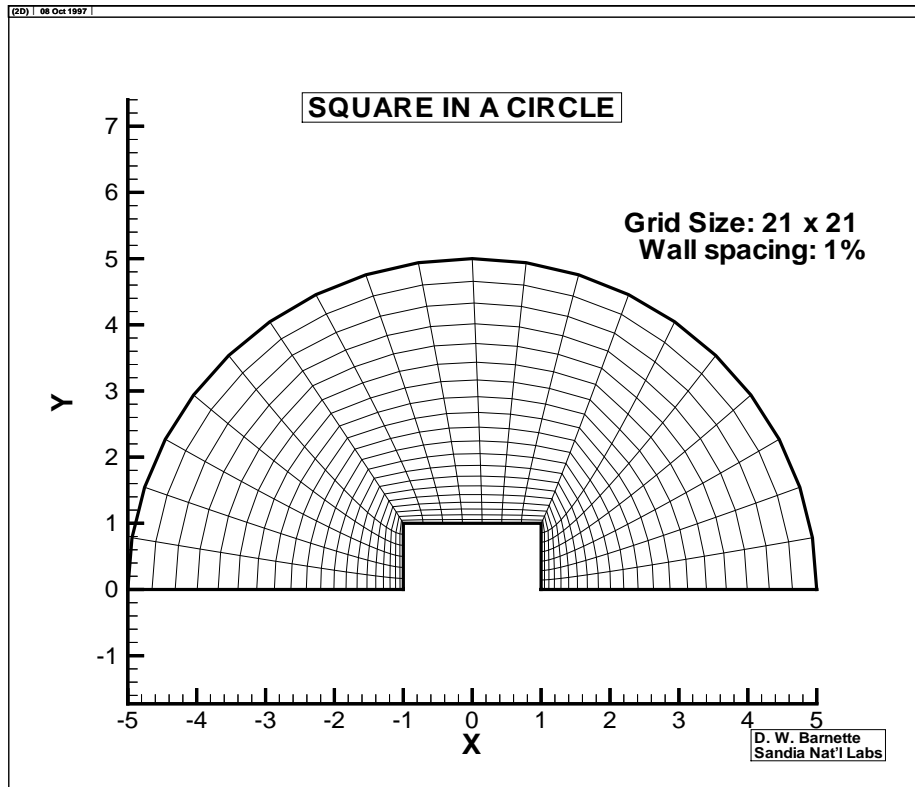


a) entire grid

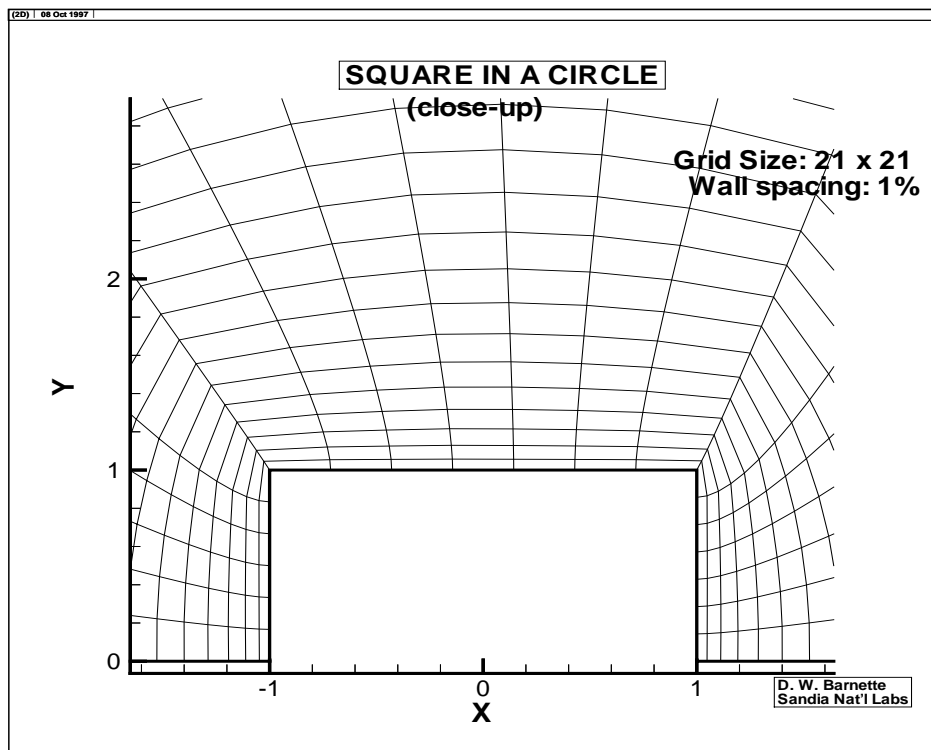


b) close-up

Figure 2. Example of a clustered algebraic grid between two concentric circles with radial grid lines normal to the inner circle.



a) entire grid



b) close-up

Figure 3. Example of a clustered algebraic grid between a half-square and a semi-circle with radial grid lines normal to the half-square.

Appendix

Derivation of Governing Grid Equations for $T_{E2}=0$

Recall Eq. 28, repeated here for convenience as

$$x_\xi^2 f_{xx} - y_\xi^2 f_{yy} = 0. \quad (A.1)$$

Let $f=u$. Upon transforming f_{xx} and f_{yy} to computational space, Eq. A.1 becomes

$$\begin{aligned} & x_\xi^2 \left\{ y_\eta \left[\left(x_\xi^y y_\eta - x_\eta^y y_\xi \right) \left(u_{\xi\xi}^y y_\eta + u_{\xi\xi}^y y_\eta - u_{\xi\eta}^y y_\xi - u_{\eta\xi}^y y_\xi \right) \right. \right. \\ & \quad \left. \left. - \left(u_{\xi\xi}^y y_\eta - u_{\eta\xi}^y y_\xi \right) \left(x_\xi^y y_\eta + x_{\xi\xi}^y y_\eta - x_\eta^y y_\xi - x_{\xi\eta}^y y_\xi \right) \right] \right. \\ & - y_\xi^2 \left[\left(x_\xi^y y_\eta - x_\eta^y y_\xi \right) \left(u_{\xi\eta}^y y_\eta + u_{\xi\eta}^y y_\eta - u_{\eta\eta}^y y_\xi - u_{\eta\xi}^y y_\xi \right) \right. \\ & \quad \left. \left. - \left(u_{\xi\eta}^y y_\eta - u_{\eta\xi}^y y_\xi \right) \left(x_\xi^y y_\eta + x_{\xi\eta}^y y_\eta - x_\eta^y y_\xi - x_{\eta\eta}^y y_\xi \right) \right] \right\} \\ & - y_\xi^2 \left\{ -x_\eta \left[\left(x_\xi^y y_\eta - x_\eta^y y_\xi \right) \left(u_{\eta\xi}^x y_\eta + u_{\eta\xi}^x y_\eta - u_{\xi\xi}^x y_\eta - u_{\xi\eta}^x y_\eta \right) \right. \right. \\ & \quad \left. \left. + \left(u_{\xi\eta}^x y_\eta - u_{\eta\xi}^x y_\eta \right) \left(x_\xi^y y_\eta + x_{\xi\xi}^y y_\eta - x_\eta^y y_\xi - x_{\xi\eta}^y y_\xi \right) \right] \right. \\ & + x_\xi^2 \left[\left(x_\xi^y y_\eta - x_\eta^y y_\xi \right) \left(u_{\eta\eta}^x y_\eta + u_{\eta\eta}^x y_\eta - u_{\xi\eta}^x y_\eta - u_{\xi\eta}^x y_\eta \right) \right. \\ & \quad \left. \left. + \left(u_{\xi\eta}^x y_\eta - u_{\eta\xi}^x y_\eta \right) \left(x_\xi^y y_\eta + x_{\xi\eta}^y y_\eta - x_\eta^y y_\xi - x_{\eta\eta}^y y_\xi \right) \right] \right\} = 0. \quad (A.2) \end{aligned}$$

This equation is regrouped in terms of coefficients of the second derivative metrics which gives

$$\left[A_1 \right] x_{\xi\xi} + \left[B_1 \right] x_{\xi\eta} + \left[C_1 \right] x_{\eta\eta} + \left[D_1 \right] y_{\xi\xi} + \left[E_1 \right] y_{\xi\eta} + \left[F_1 \right] y_{\eta\eta} = RHS_1 \quad (A.3)$$

where

$$\begin{aligned} A_1 &= -x_\xi^2 y_\eta^2 \left(u_{\xi\xi}^y y_\eta - u_{\eta\xi}^y y_\xi \right) + y_\xi^2 x_\eta^2 \left(x_\xi^y y_\eta - x_\eta^y y_\xi \right) u_\eta + y_\xi^2 x_\eta^2 y_\eta \left(u_{\xi\xi}^x y_\eta - u_{\eta\xi}^x y_\xi \right) \\ &= u_\xi \left(-x_\xi^2 y_\eta^2 y_\eta + y_\xi^2 x_\eta^2 y_\eta \right) + u_\eta \left(x_\xi^2 y_\eta^2 y_\xi + y_\xi^2 x_\eta^2 x_\xi y_\eta - y_\xi^3 x_\eta^2 - y_\xi^2 x_\eta^2 x_\xi y_\eta \right) \\ &= u_\xi y_\eta \left(-x_\xi^2 y_\eta^2 + y_\xi^2 x_\eta^2 \right) + u_\eta y_\xi \left(x_\xi^2 y_\eta^2 - y_\xi^2 x_\eta^2 \right) \\ &= 0 \quad (\text{via Eq. 31}) \end{aligned} \quad (A.4)$$

$$\begin{aligned}
B_1 &= x_\xi^2 y \eta y_\xi \left(u_\xi y \eta - u_\eta y_\xi \right) + x_\xi^2 y_\xi y \eta \left(u_\xi y \eta - u_\eta y_\xi \right) - y_\xi^2 x \eta \left(x_\xi y \eta - x_\eta y_\xi \right) u_\xi \\
&\quad - y_\xi^3 x \eta \left(u_\xi x \eta - u_\eta x_\xi \right) - y_\xi^2 x_\xi \left(x_\xi y \eta - x_\eta y_\xi \right) u_\eta - y_\xi^2 x_\xi y \eta \left(u_\xi x \eta - u_\eta x_\xi \right) \\
&= u_\xi \left(2x_\xi^2 y \eta^2 y_\xi - 2y_\xi^2 x_\xi x \eta y \eta \right) - u_\eta \left(2x_\xi^2 y_\xi^2 y \eta - 2y_\xi^3 x_\xi x \eta \right) \\
&= 2u_\xi x_\xi y_\xi y \eta \left(x_\xi y \eta - y_\xi x \eta \right) - 2u_\eta y_\xi^2 x_\xi \left(x_\xi y \eta - y_\xi x \eta \right) \\
&= 2J x_\xi y_\xi \left(u_\xi y \eta - u_\eta y_\xi \right)
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
C_1 &= -x_\xi^2 y_\xi^2 \left(u_\xi y \eta - u_\eta y_\xi \right) + y_\xi^2 x_\xi \left(x_\xi y \eta - x_\eta y_\xi \right) u_\xi + y_\xi^3 x_\xi \left(u_\xi x \eta - u_\eta x_\xi \right) \\
&= u_\eta \left(x_\xi^2 y_\xi^3 - y_\xi^3 x_\xi^2 \right) + u_\xi \left(-x_\xi^2 y_\xi^2 y \eta + y_\xi^2 x_\xi^2 y \eta - y_\xi^3 x_\xi x_\eta + y_\xi^3 x_\xi x \eta \right) \\
&= 0 \quad \text{identically}
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
D_1 &= -x_\xi^2 y \eta \left(x_\xi y \eta - x_\eta y_\xi \right) u_\eta + x_\xi^2 y \eta x \eta \left(u_\xi y \eta - u_\eta y_\xi \right) - y_\xi^2 x_\eta^2 \left(u_\xi x \eta - u_\eta x_\xi \right) \\
&= u_\eta \left(-x_\xi^3 y_\eta^2 + x_\xi^2 y \eta x \eta y_\xi - x_\xi^2 y \eta x \eta y_\xi + y_\xi^2 x_\eta^2 x_\xi \right) + u_\xi \left(x_\xi^2 y_\eta^2 x \eta - y_\xi^2 x_\eta^3 \right) \\
&= u_\eta x_\xi \left(-x_\xi^2 y_\eta^2 + y_\xi^2 x_\eta^2 \right) + u_\xi x_\eta \left(x_\xi^2 y_\eta^2 - y_\xi^2 x_\eta^2 \right) \\
&= 0 \quad (\text{via Eq. 31})
\end{aligned} \tag{A.7}$$

$$E_1 = x_\xi^2 y \eta \left(x_\xi y \eta - x_\eta y_\xi \right) u_\xi - x_\xi^3 y \eta \left(u_\xi y \eta - u_\eta y_\xi \right) + x_\xi^2 y_\xi \left(x_\xi y \eta - x_\eta y_\xi \right) u_\eta$$

$$\begin{aligned}
& -x_{\xi}^2 y_{\xi} x_{\eta} \left(u_{\xi} y_{\eta} - u_{\eta} y_{\xi} \right) + y_{\xi}^2 x_{\eta} x_{\xi} \left(u_{\xi} x_{\eta} - u_{\eta} x_{\xi} \right) + y_{\xi}^2 x_{\xi} x_{\eta} \left(u_{\xi} x_{\eta} - u_{\eta} x_{\xi} \right) \\
& = u_{\xi} \left(-2x_{\xi}^2 x_{\eta} y_{\xi} y_{\eta} + 2x_{\eta}^2 y_{\xi}^2 x_{\xi} \right) + u_{\eta} \left(2x_{\xi}^2 y_{\xi} y_{\eta} - 2x_{\xi}^2 y_{\xi}^2 x_{\eta} \right) \\
& = -2u_{\xi} x_{\xi} x_{\eta} y_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) + 2u_{\eta} x_{\xi}^2 y_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) \\
& = 2J x_{\xi} y_{\xi} \left(u_{\eta} x_{\xi} - u_{\xi} x_{\eta} \right)
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
F_1 & = -x_{\xi}^2 y_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi} + x_{\xi}^3 y_{\xi} \left(u_{\xi} y_{\eta} - u_{\eta} y_{\xi} \right) - x_{\xi}^2 y_{\xi}^2 \left(u_{\xi} x_{\eta} - u_{\eta} x_{\xi} \right) \\
& = u_{\xi} \left(-x_{\xi}^3 y_{\xi} y_{\eta} + x_{\xi}^2 y_{\xi}^2 x_{\eta} + x_{\xi}^3 y_{\xi} y_{\eta} - x_{\xi}^2 y_{\xi}^2 x_{\eta} \right) + u_{\eta} \left(-x_{\xi}^3 y_{\xi}^2 + x_{\xi}^3 y_{\xi}^2 \right) \\
& = 0 \quad (\text{identically})
\end{aligned} \tag{A.9}$$

and

$$\begin{aligned}
RHS_1 & = -x_{\xi}^2 y_{\xi}^2 \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\xi} + x_{\xi}^2 y_{\eta} y_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\eta} + x_{\xi}^2 y_{\xi} y_{\eta} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\eta} \\
& \quad - x_{\xi}^2 y_{\xi}^2 \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\eta\eta} + y_{\xi}^2 x_{\eta}^2 \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\xi} - y_{\xi}^2 x_{\eta} x_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\eta} \\
& \quad - y_{\xi}^2 x_{\xi} x_{\eta} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\xi\eta} + y_{\xi}^2 x_{\xi}^2 \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) u_{\eta\eta} \\
& = u_{\xi\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) \left(x_{\eta}^2 y_{\xi}^2 - x_{\xi}^2 y_{\eta}^2 \right) + u_{\eta\eta} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) \left(x_{\xi}^2 y_{\xi}^2 - x_{\xi}^2 y_{\xi}^2 \right) \\
& \quad + u_{\xi\eta} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) \left(x_{\xi}^2 y_{\eta} y_{\xi} + x_{\xi}^2 y_{\xi} y_{\eta} - y_{\xi}^2 x_{\xi} x_{\eta} - y_{\xi}^2 x_{\xi} x_{\eta} \right) \\
& = 2u_{\xi\eta} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) \left(x_{\xi}^2 y_{\xi} y_{\eta} - y_{\xi}^2 x_{\xi} x_{\eta} \right) \\
& = 2 \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right)^2 x_{\xi} y_{\xi} u_{\xi\eta}
\end{aligned}$$

$$= 2J^2 x_{\xi}^y y_{\xi}^u u_{\xi\eta} . \quad (\text{A.10})$$

Substitution of the values for the coefficients into Eq. A.3 results in

$$2Jx_{\xi}^y y_{\xi}^u \left(u_{\xi}^y \eta - u_{\eta}^y \xi \right) x_{\xi\eta} + 2Jx_{\xi}^y y_{\xi}^u \left(u_{\eta}^x \xi - u_{\xi}^x \eta \right) y_{\xi\eta} = 2J^2 x_{\xi}^y y_{\xi}^u u_{\xi\eta} \quad (\text{A.11})$$

or

$$\left(u_{\xi}^y \eta - u_{\eta}^y \xi \right) x_{\xi\eta} + \left(u_{\eta}^x \xi - u_{\xi}^x \eta \right) y_{\xi\eta} = Ju_{\xi\eta} . \quad (\text{A.12})$$

This equation can be written in a more compact notation by writing the terms in parentheses in physical rather than computational space. Recall the equations for the two-dimensional transformation metrics[1,2]

$$\xi_x = \frac{y\eta}{J} \quad \xi_y = -\frac{x\eta}{J} \quad \eta_x = -\frac{y\xi}{J} \quad \eta_y = \frac{x\xi}{J} . \quad (\text{A.13})$$

Then

$$Ju_x = J \left(u_{\xi}^x \xi_x + u_{\eta}^x \eta_x \right) = J \left(\frac{u_{\xi}^y \eta - u_{\eta}^y \xi}{J} \right) = u_{\xi}^y \eta - u_{\eta}^y \xi \quad (\text{A.14})$$

and

$$Ju_y = J \left(u_{\xi}^x \xi_y + u_{\eta}^x \eta_y \right) = J \left(\frac{-u_{\xi}^x \eta + u_{\eta}^x \xi}{J} \right) = u_{\eta}^x \xi - u_{\xi}^x \eta . \quad (\text{A.15})$$

This allows Eq. A.12 to be written as

$$u_x^x \xi_{\eta} + u_y^y \xi_{\eta} = u_{\xi}^x . \quad (\text{A.16})$$

The analogous result for Eq. A.1 with $f=v$ may be written as

$$v_x^x \xi_{\eta} + v_y^y \xi_{\eta} = v_{\xi}^x . \quad (\text{A.17})$$

The last two equations may be written in matrix form as

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{Bmatrix} x_{\xi\eta} \\ y_{\xi\eta} \end{Bmatrix} = \begin{Bmatrix} u_{\xi\eta} \\ v_{\xi\eta} \end{Bmatrix}. \quad (\text{A.18})$$

If the determinant of the matrix is non-zero, we may solve for the metric derivatives and obtain

$$x_{\xi\eta} = \frac{v_y u_{\xi\eta} - u_y v_{\xi\eta}}{v_y u_x - v_x u_y} \quad (\text{A.19})$$

and

$$y_{\xi\eta} = \frac{u_x v_{\xi\eta} - v_x u_{\xi\eta}}{v_y u_x - v_x u_y}. \quad (\text{A.20})$$

Transforming the velocity derivatives back into computational space, one may write the above equations as

$$x_{\xi\eta} = J \frac{\left(v_{\eta} x_{\xi} - v_{\xi} x_{\eta} \right) u_{\xi\eta} - \left(u_{\eta} x_{\xi} - u_{\xi} x_{\eta} \right) v_{\xi\eta}}{\left(v_{\eta} x_{\xi} - v_{\xi} x_{\eta} \right) \left(u_{\xi} y_{\eta} - u_{\eta} y_{\xi} \right) - \left(v_{\xi} y_{\eta} - v_{\eta} y_{\xi} \right) \left(u_{\eta} x_{\xi} - u_{\xi} x_{\eta} \right)}. \quad (\text{A.21})$$

Multiplying and collecting terms, the denominator becomes

$$\begin{aligned} & \left(v_{\eta} x_{\xi} - v_{\xi} x_{\eta} \right) \left(u_{\xi} y_{\eta} - u_{\eta} y_{\xi} \right) - \left(v_{\xi} y_{\eta} - v_{\eta} y_{\xi} \right) \left(u_{\eta} x_{\xi} - u_{\xi} x_{\eta} \right) \\ &= v_{\eta} u_{\xi} \left(x_{\xi} y_{\eta} - x_{\eta} y_{\xi} \right) + v_{\xi} u_{\xi} \left(x_{\eta} y_{\eta} - x_{\eta} y_{\eta} \right) + v_{\eta} u_{\eta} \left(x_{\xi} y_{\xi} - x_{\xi} y_{\xi} \right) + v_{\xi} u_{\eta} \left(x_{\eta} y_{\xi} - y_{\eta} x_{\xi} \right) \\ &= J \left(v_{\eta} u_{\xi} - v_{\xi} u_{\eta} \right). \end{aligned} \quad (\text{A.22})$$

Eqs. A.19 and A.20 may now be written in their final form as

$$x_{\xi\eta} = \left(\frac{v_{\eta} u_{\xi\eta} - u_{\eta} v_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}} \right) x_{\xi} + \left(\frac{u_{\xi} v_{\xi\eta} - v_{\xi} u_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}} \right) x_{\eta} \quad (\text{A.23})$$

and

$$y_{\xi\eta} = \left(\frac{v_{\eta} u_{\xi\eta} - u_{\eta} v_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}} \right) y_{\xi} + \left(\frac{u_{\xi} v_{\xi\eta} - v_{\xi} u_{\xi\eta}}{v_{\eta} u_{\xi} - u_{\eta} v_{\xi}} \right) y_{\eta}. \quad (\text{A.24})$$

These are identical to Eqs. 37 and 38 in the text.

EXTERNAL DISTRIBUTION:

5	Micro Craft Attn: J. A. Benek (1) N. Suhs (4) 207 Big Springs Avenue P.O. Box 370 Tullahoma, TN 37388-0370	1	Dr. Lori Frietag Argonne National Laboratory 9700 S. Cass Ave, MCS221 Argonne, IL 60439-4844
1	Dr. C. Wayne Mastin Nichols Research 2524 South I-20 Frontage Rd. Suite A P.O. Box 820186 Vicksburg, MS 39182	1	Dr. Ed D'Azevedo Oak Ridge National Laboratory PO Box 2008, Bldg. 6012, MS 6367 Oak Ridge, TN 37831-6367
1	Prof. Timothy Baker Dept. of Mechanical & Aerospace Engineering Princeton University Princeton, NJ 08544	1	Prof. Martin Berzins University of Utah Computer Science, MEB Building Salt Lake City, UT 84112
1	Dr. Ray Gomez Advanced Programs Office Mail Code EG3 NASA Johnson Space Center Houston, TX 77058	1	Charles Nietubicz, Director Major Shared Resource Center US Army Research Laboratory Aberdeen Proving Ground, MD 21005-5067
1	Prof. David Whitfield Engineering Research Center Mississippi State University P.O. Box 9627 Mississippi State, MS 39762	2	Dr. Michelle Hribar Dr. Jerry C. Yan MS T27A-2 NASA Ames Research Center Moffett Field, CA 94035-1000
1	Dr. Bob Meakin NASA Ames Research Center MS 258-1 Moffett Field, CA 94035-1000	1	Dr. Pieter Buning Configuration Aerodynamics Branch MS 499 NASA Langley Research Center Hampton, VA 23681-0001
5	Lockheed-Martin Skunkworks Attn: G. Shrewsbury (4) J. Vadyak (1) Computational Aerodynamics 1011 Lockheed Way Palmdale, CA 93599-2523	1	CFD Research Corporation Attn: Sami D. Habchi 215 Wynn Drive Huntsville, AL 35805
1	Mr. D. Howlett Engineering Chief Aerodynamics & Computational Fluid Dynamics Lockheed Martin Tactical Aircraft Systems PO Box 748 Mail Zone 9333 Ft. Worth, TX 76101		

INTERNAL DISTRIBUTION:

1	MS	0188	D. Chavez, 4523
1		0318	G. S. Davison, 9215
1		0825	B. Hassan, 9115
1		0825	J. Payne, 9115
1		0825	F. Blottner, 9115
1		0441	P. Knupp, 9226
1		0825	W. Rutledge, 9115
1		0321	W. J. Camp, 9200
1		1111	S. S. Dosanjh, 9221
10		1111	D. W. Barnette, 9221
1		0318	G. S. Davison, 9215
1		1110	D. E. Womble, 9222
1		1110	D. Greenburg, 9223
1		1109	A. L. Hale, 9224
1		1111	G. Heffelfinger, 9225
1		0441	R. W. Leland, 9226
1		0819	J. Peery, 9231
1		0820	P. Yarrington, 9232
1		0439	D. R. Martinez, 9234
1		1166	J. D. Kotulski, 9352
1		1166	D. J. Riley, 9352
1		1166	D. C. Turner, 9352
1		9018	Central Technical Files, 8940-2
2		0899	Technical Library, 4619
2		0619	Review & Approval Desk, 12690
			For DOE/OSTI